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THE ROLE OF A LENGTH CONSTRAINT IN THE DESIGN OF MINIMUM-DRAG BODIES†

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The key role of a constraint on the length of a designed body is demonstrated by taking the example of symmetric profiles which achieve minimum wave drag in a supersonic flow. As a result of this an optimal body can contain a rear-base which emerges as a section of a boundary extremum. By assumption, there is no gas flow around the rear base, and the "base" pressure p^+ which acts on them is specified and is independent of the form of the required contour and the ordinate y. When designing profiles, in addition to their length, it is customary to specify the area of the longitudinal section F and other isoperimetric conditions. Even when $p^+ = 0$, which, naturally, does not occur, it is now necessary to introduce a rear base for extremely small F. When $p^+ > 0$, a base appears earlier still. The replacement of the optimal contours with a rear base by a "pseudo-optimal" contours with a sharp edge leads to an increase in drag of tens and hundreds of percent. Special attention has been paid to cases in which p^+, due to heat supply in the base domain, for example, exceeds the free-stream pressure. Here, there is always a rear base and, when $F < F_0$, where F_0 depends on p^+/p_∞, the form of the optimal contours is the same as in the problem without a specified F. In this case, the optimal configuration is a hollow or partially hollow "checkmark".

The existence of a front face in the case of axially symmetric forebodies of minimum drag had already been established by Newton. The need for such a face in the case of sufficiently thick foresections of two-dimensional bodies has been demonstrated in [1]. In the case of the optimal forebodies of fixed volume, a constraint on the maximum magnitude of the axial coordinate (if this constant if not imposed, a constraint on its minimum magnitude and, consequently, a constraint on the length of the forebody) leads to the appearance of a face of a different type: a front part of the specified body with a spike projecting from it [2]. In the design of optimal nozzles and afterbodies, the rear bases, which are sections of a boundary extremum, were introduced in [3].

While in the design of fore- and afterbodies and nozzles, it became customary long ago for front faces and rear bases to be an accessory of the optimal contours, this was far from being the case in problems concerned with the design of closed bodies. Up to the present in such problems, rear bases, if they were introduced at all, were introduced as physically expedient, rather than as sections of a boundary extremum. In this respect, the paper [5] and the collection of papers [6] are significant. The greatest advance in this direction was perhaps made in the first of these two publications. In [5], in the optimal design of thin symmetric profiles within the framework of the linear theory of supersonic flows, the size of a rear base and the condition for its appearance were obtained from the solution of a variational problem. However, the conditions for the optimality of the rear base as a section of a boundary extremum were not written out or discussed, and its introduction following the predecessors cited in [5] was explained by physical arguments. In the paper [7] from the collection [6], while reference is made to [5], the possibility of the appearance of a rear base in the same problem is simply ignored.

In other papers in [6], a rear base appeared in the case of bodies which achieve a minimum wave drag coefficient C_x within the framework of Newton's drag law. In this case, the pressure p^+ which acts on the shaded part of the body was taken as being equal to p_∞ in accordance with Newton's formula without any stipulations. Simultaneously, there was no mention anywhere of the very existence of a rear base which, together with the absence of a term with the pressure $p^+ = p_\infty$ acting on the base in the formula for C_x, almost completely masked its presence. As a result, it is difficult for the reader to note that the discussion is about the whole body and not about its forebody. In the case of such masking of the existence of rear bases, they cannot be discussed either as sections of a boundary extremum or even regarding their physical advisability. Under such conditions, errors are unavoidable, by virtue of which a rear base appeared when there are no constraints on the body length as in [8].

In the examples we have discussed above, faces and bases were present in variational problems of supersonic aerodynamics. Isolated from these, there are problems concerned with the design of plane and axially symmetric bodies, of specified length and area F or volume in a subsonic flow of an ideal

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(inviscid and non-heat-conducting) gas with a maximum critical Mach number M_∞^*. When $M_\infty \leqslant M_\infty^*$, their $C_x = 0$. It has been established using comparison theorems [9, 10] that the contours of such bodies consist of leading face and rear base, and a sonic streamline which joins them. These face and base are segments of a boundary extremum which appear due to a constraint on the length. Within the framework of a model of unseparated flow around bodies, the pressure and the other parameters on them are variable and are found together with the solution of the whole design problem.

The following order of presentation is adopted below. A variational problem, which is subsequently solved, is formulated using certain local models in Section 1. In Section 2, the necessary optimality conditions are obtained for this problem, including the optimality conditions of the rear base as a segment of a boundary extremum. The conditions of Section 2 are used in Section 3 to design profiles in the case of a single isoperimetric condition, a specified area for the longitudinal cross-section. Here, the advantages of optimal bodies with rear base, which are designed using local models, is confirmed by computations in the approximation of the Euler equations. In Section 4, bodies of the shape of hollow and partially hollow "check mark" and bodies with screens, which are optimal when $p^+ > p_\infty$ and have not been considered earlier, are constructed. In Section 5, the errors and inaccuracies in a number of papers, where the authors also introduced rear bases when constructing optimal bodies, are corrected. In concluding (Section 6), arguments are presented concerning the importance of rear bases when constructing minimum drag bodies in a flow of a viscous gas (or liquid).

1. FORMULATION OF THE VARIATIONAL PROBLEM

Suppose that x and y are Cartesian coordinates. The x axis is directed along the vector of the supersonic free-stream velocity $\mathbf{V}_\infty$, the origin of the system of coordinates is associated with the leading point of the body i, and the specified length of the body l is adopted as the scale of length. Then, at the end point f of the upper contour of a body which is symmetrical about the x axis and has a fine-pointed rear edge (Fig. 1a), $x_f = 1, y_f = 0$. Henceforth subscripts $i, \ldots$ are assigned to parameters at the points $i, \ldots$, and ∞ to the free-stream parameters. Similarly (Fig. 1b), when there is a rear base $f^\circ f$, we have $x \equiv 1$ in this base. It has already been mentioned that a base pressure p^+, which is independent of the contour shape if° and the y ordinate, acts on the rear base. If p^+ and the pressure p at any point if° are referred to $\rho_\infty V_\infty^2$, where ρ is the gas density and $V = |\mathbf{V}|$, then, for the wave drag coefficient, we have

$$C_x = \int\limits_{y_i=0}^{y_{f^\circ}} p dy - p^+ y_{f^\circ} \tag{1.1}$$

with a pressure on the profile, p, which depends on the contour shape if°. Henceforth, C_x and other integral characteristics are defined for a half of the profile.

For the purposes of this paper it is sufficient to restrict the treatment to local models of the flow around a body. In the case of such models, the pressure at any point of the contour if° for a specified

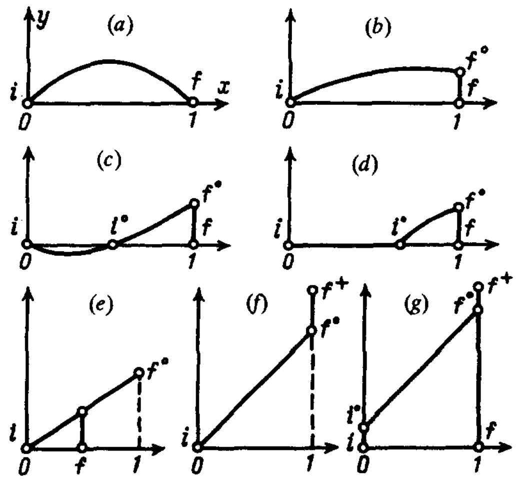

Fig. 1.

free stream is a known function of the angle ϑ between the tangent to the body and the x axis, that is, $p = p(\vartheta)$. If $x = x(y)$ is the equation of $if°$ and $x' = dx/dy$, then, according to the definition of ϑ

$$E \equiv x' - \operatorname{ctg}\vartheta = 0 \tag{1.2}$$

When Newton's drag, which holds when $\vartheta \leqslant \pi/2$, is used, in accordance with the method which has been adopted for reducing the pressure to dimensionless form, we have

$$p(\vartheta) = p_\infty + 1/2(1 + \operatorname{sign}\vartheta)\sin^2\vartheta \tag{1.3}$$

In the case of a perfect gas $p_\infty = 1/(\varkappa M_\infty^2)$, where $\varkappa$ is the adiabatic exponent. The constraint $\vartheta \leqslant \pi/2$ precludes recesses, by the use of which it should be possible to reduce C_x to zero, on the windward side of the body.

Another local model is obtained as a combination of Newton's drag law (1.3) when $0 \leqslant \vartheta \leqslant \pi/2$ and the solution for the simple rarefaction wave adjoining the uniform free stream when $\vartheta \leqslant 0$

$$p(\vartheta) = \begin{cases} p_\infty + \sin^2\vartheta & \text{when} \quad 0 \leqslant \vartheta \leqslant \pi/2 \\ P(\vartheta) & \text{when} \quad \vartheta \leqslant 0 \end{cases} \tag{1.4}$$

$P(\vartheta)$ is determined using the formulae for a simple wave. If s is the specific entropy of the gas, $h = h(p, s)$, the specific enthalpy, is a known function of p and s, H is the total enthalpy and $\alpha = \arcsin(1/M)$ is the Mach angle, then such a simple wave is described by the relations [11]

$$2h[P(\vartheta),\ s_\infty] + V^2 = 2H = \frac{2}{(\varkappa - 1)M_\infty^2} + 1, \quad \frac{1}{\rho} = \left(\frac{\partial h}{\partial p}\right)_s$$

$$I(\vartheta,\ p) \equiv \vartheta - \int_{p_\infty}^{P(\vartheta)} \frac{dp}{A(p)} = \vartheta - \frac{\pi}{2} + \alpha + \sqrt{\frac{\varkappa+1}{\varkappa-1}}\operatorname{arctg}\left(\sqrt{\frac{\varkappa-1}{\varkappa+1}}\operatorname{ctg}\alpha\right) = I_\infty = 0 \tag{1.5}$$

$$A(p) = \frac{\rho V^2}{\sqrt{M^2 - 1}} \equiv \frac{\rho V^2}{\operatorname{ctg}\alpha}, \quad M^2 = \frac{V^2}{a^2}, \quad \frac{1}{a^2} = \left(\frac{\partial \rho}{\partial p}\right)_s = -\rho^2\left(\frac{\partial^2 h}{\partial p^2}\right)_s$$

Here I is the Riemann invariant which corresponds to the C^- characteristics; the second formulae for $2H$ and I are written for a perfect gas, and ρ and a, which are expressed in terms of the derivatives of h and h, are known functions of p and s. Relations (1.5) gives the implicit dependence $p = P(\vartheta)$. This dependence is a one-to-one dependence since, by virtue of the formula for ϑ

$$p_\vartheta \equiv \frac{dP(\vartheta)}{d\vartheta} = A(p) \geqslant 0 \tag{1.6}$$

where the equality only applies in the case of $M \to \infty$ when $V^2 \to 2H$ and $\rho \to 0$. According to (1.5), $P(0) = p_\infty$ and, consequently, the function $p(\vartheta)$, is defined by equalities (1.4) when $\vartheta = 0$ is continuous with discontinuous first and higher derivatives with respect to ϑ. In accordance with inequality (1.6), $p(\vartheta) < p_\infty$ in the case of negative ϑ, which ensures the somewhat higher accuracy of model (1.4) compared with the initial Newtonian model (1.3). In obtaining formulae (1.5), no account has been taken of the non-isentropic character of the flow and the fact that the invariant I differs from $I_\infty = 0$. If the bow-wave, which is detached from the sharpened leading edge of the body, is weak, then, as is well known [11, 12], the increments in s and I on passing across it are quantities of the order of the cube of the pressure increment. Therefore, in the case of comparatively slender bodies as well as in the case of low supersonic free-stream Mach numbers (subject to the condition that the flow downstream of the low-wave is supersonic), the local model

$$p(\vartheta) = P(\vartheta) \tag{1.7}$$

with $P(\vartheta)$ from (1.5) is valid with an accuracy up to ϑ^2 inclusive for the whole contour $if°$. The linearization (1.5)–(1.7), which presupposes that ϑ^2 is small, also gives the well-known formula of linear theory

$$p = p_\infty + \frac{\vartheta}{\sqrt{M_\infty^2 - 1}} \tag{1.8}$$

but which is now only valid up to ϑ inclusive.

Here, as everywhere above, the pressure has been made dimensionless by reference to $\rho_\infty V_\infty^2$.

If only the length of the body is specified and $p^+ \leqslant p_\infty$, then the solution of the problem of constructing the contour if, which reaches the minimum C_x is trivial. In the approximation of any of the local models which have been described above, as well as within the framework of the complete system of Euler equations, it is given by the segment $0 \leqslant x \leqslant 1$ of the x axis, that is, the body of minimum wave drag is a plate with $C_x = 0$.

Suppose that, together with the specification of the length, the body being profiled must also satisfy N isoperimetric conditions, in particular, of a geometric nature, which are written in the form

$$F^n = \int_{y_i=0}^{y_{f^\circ}} \Phi^n(p,\ \vartheta,\ x,\ y)dy - \int_{y_f=0}^{y_{f^\circ}} \varphi^n(p^+,\ x,\ y)dy, \quad n = 1,\ \ldots,\ N \tag{1.9}$$

Here, F^n are specified constants and Φ^n and φ^n are known functions of their arguments. Of course, φ^n may also depend on ϑ but, for the purposes of this investigation, it is sufficient to restrict the treatment to $\varphi^n(p^+, x, y)$. What is far more important is that, in (1.9) and also in (1.1), it is necessary to include integrals over a rear base $f^\circ f$ which is possible but is not always present in the optimal configuration, and, moreover, x should not be replaced by unity when writing the isoperimetric conditions in such integrals.

2. NECESSARY CONDITIONS FOR OPTIMALITY

In order to obtain the necessary conditions for optimality, we construct the Lagrange functional

$$I = \int_{y_i=0}^{y_{f^\circ}} [\Phi(p,\ \vartheta,\ x,\ y,\ \boldsymbol{\mu}) + \lambda E]dy - \int_{y_f=0}^{y_{f^\circ}} \varphi(p^+,\ x,\ y,\ \boldsymbol{\mu})dy$$

$$\Phi(p,\ \vartheta,\ x,\ y,\ \boldsymbol{\mu}) = p(\vartheta) + \sum_{n=1}^{N} \mu^n \Phi^n(p,\ \vartheta,\ x,\ y)$$

$$\varphi(p^+,\ x,\ y,\ \boldsymbol{\mu}) = p^+ + \sum_{n=1}^{N} \mu^n \varphi^n(p^+,\ x,\ y)$$

in which μ^n are constant and $\lambda = \lambda(y)$ are variable undetermined Lagrange multipliers, $\boldsymbol{\mu}$ is an N-dimensional vector with components μ^n and E is the left-hand side of Eq. (1.2). We shall call any contour, which gives a body of the specified length with a rear base $f^\circ f$ or without it and which satisfies conditions (1.2) and (1.9), a permissible contour. In the case of a variation when the initial contour (which is not necessarily the optimal) and the varied contour generatrix are permissible (other ways of variation are, naturally, not considered), the variations of I and C_x are identical. In the common case, the optimal contour of a body of fixed length may include not only a rear base but also a front face which must be stipulated when writing out the expressions for C_x, conditions (1.9) and the Lagrange functional.

Taking the above and the expressions for I and E into account, for a permissible variation of a contour if which is sharpened at point i and either is sharpened at point f or has a rear base, we have

$$\delta C_x = \delta I = X^{f^\circ} \Delta x_{f^\circ} + Y^{f^\circ} \Delta y_{f^\circ} + \int_{y_i=0}^{y_{f^\circ}} (A^x \delta x + A^\vartheta \delta\vartheta)dy - \int_{y_f=0}^{y_{f^\circ}} A^+ \delta x dy \tag{2.1}$$

Here, Δx_{f° and Δy_{f° are the increments in the coordinates of the point f° and, in the absence of a rear base, when $f^\circ \equiv f$, of the point f; δx and $\delta\vartheta$ are the variations, that is, the increments of x and ϑ for the varied and the initial contours in the case of a fixed ordinate y; $X^{f^\circ}, Y^{f^\circ}, A^x, A^\vartheta$ and A^+ are known functions of x, y, $p = p(\vartheta)$, ϑ, p^+ and the Lagrange multipliers at the corresponding points. Henceforth, f° is replaced by f in the case of bodies without a rear base. By virtue of this, the last integral in (2.1) is not present in the case of such bodies.

On varying any permissible contour, using an arbitrary choice of the variable Lagrange multiplier λ, the coefficient A^{ϑ} in if° can vanish. When account is taken of the expression for A^{ϑ}, this gives the finite equation

$$\lambda = -(\Phi_{\vartheta} + \Phi_p p_{\vartheta})\sin^2\vartheta, \quad \Phi_{\vartheta} = \left(\frac{\partial\Phi}{\partial\vartheta}\right)_{p,x,y}, \quad \Phi_p = \left(\frac{\partial\Phi}{\partial p}\right)_{\vartheta,x,y} \tag{2.2}$$

for determining λ in which $p_{\vartheta} = dp/d\vartheta$ is found in accordance with the local model employed.

After this, only terms which are proportional to $\Delta x_{f^{\circ}}$ and $\Delta y_{f^{\circ}}$ and the integrals from the variations in δx in if° and $f^{\circ}f$ remain in (2.1). It is clear that the integral over $f^{\circ}f$ is in (2.1) only if the initial contour has a rear base. When there are N isoperimetric conditions, the above-mentioned increments and variations are not independent. Their independence is achieved by introducing N "compensating" points k_n in the segment if° of the contour of the body and by determining the constant Lagrange multipliers $\mu^1, \ldots, \mu^N$ from the linear system obtained, if one puts, at the point k_n

$$A^x \equiv \Phi_x - \lambda' = 0, \quad \Phi_x = \left(\frac{\partial\Phi}{\partial x}\right)_{p,\vartheta,y} \tag{2.3}$$

with the derivative which has been found for the initial contour by differentiation of the right-hand side of (2.2).

Satisfying equalities (2.3) for the present just at the points k_n enables one to preserve the magnitude of all of the functionals specified in (1.9) when varying x in the neighbourhood of any point of the contour if due to the simultaneous variation in x in small neighbourhoods of the points k_n. Due to the choice of μ, a variation in x in small neighbourhoods of the points k_n introduces a contribution into the variation of C_x of a higher order than δx at the other points if and, also, into $\Delta x_{f^{\circ}}$ and $\Delta y_{f^{\circ}}$. Hence, δx, $\Delta x_{f^{\circ}}$ and $\Delta y_{f^{\circ}}$ may now be considered as being independent. Consequently, if the segment of the contour if°, in which the δx are arbitrary, realizes a minimum in C_x then equality (2.3) must be satisfied not only at the compensating points but everywhere in if°.

If the optimal body has a rear base, the permissible $\Delta y_{f^{\circ}}$ are arbitrary. When there is no rear base, the points f and f° coincide and the permissible $\Delta y_{f^{\circ}} \geqslant 0$. In addition, $\Delta x_{f^{\circ}} \leqslant 0$ in both cases. If the contour $if^{\circ}f$ is optimal, then $\delta C_x \geqslant 0$ for any permissible variation. When account is taken of the possible signs of $\Delta y_{f^{\circ}}$ and $\Delta x_{f^{\circ}}$, the optimality conditions, which must be satisfied at the point f°, which coincides or does not coincide with f, reduce to the two inequalities

$$Y^{f^{\circ}} \equiv (\Phi - \varphi - \lambda \operatorname{ctg}\vartheta)_{f^{\circ}} \geqslant 0, \quad X^{f^{\circ}} \equiv \lambda_{f^{\circ}} \leqslant 0$$

After eliminating λ using Eq. (2.2), they take the form

$$\begin{gathered} [\Phi - \varphi + (\Phi_{\vartheta} + \Phi_p p_{\vartheta})\sin\vartheta\cos\vartheta]_{f^{\circ}} \geqslant 0 \\ (\Phi_{\vartheta} + \Phi_p p_{\vartheta})_{f^{\circ}} \sin^2\vartheta_{f^{\circ}} \geqslant 0 \end{gathered} \tag{2.4}$$

where the quantities, apart from ϑ, are the limiting values on approaching the point f° from the left along if° and $\varphi_{f^{\circ}} = \varphi(p^+, 1, y_{f^{\circ}}, \mu)$.

The violation of the first inequality of (2.4) at the point $f^{\circ} \equiv f$ of a body without a rear base indicates the need for its introduction. In the case of bodies without a rear base, satisfaction of the second inequality of (2.4) indicates that C_x increases as the length of the body decreases. In the case of bodies with a rear base, the same inequality is the condition that the rear base $f^{\circ}f$ is a segment of a boundary extremum. On the rear base itself, the permissible $\delta x \leqslant 0$. Hence, the further condition of the fact that the rear base is a segment of a boundary extremum with respect to x reduces to the inequality

$$A^+ \equiv \varphi_x(p^+, 1, y, \mu) \geqslant 0 \tag{2.5}$$

On eliminating λ from (2.2) and (2.3), we arrive at an equation for determining the optimal contour (OC), that is, the function $x = x(y)$

$$[(\Phi_{\vartheta} + \Phi_p p_{\vartheta})\sin^2\vartheta]' + \Phi_x = 0 \tag{2.6}$$

The second order of this equation makes it possible (at least, in principle) to construct the segment

$if°$ of the OC which, starting from a point i with coordinates $x = y = 0$, gives a minimum of C_x. In this case, the second arbitrary parameter for a body with a rear base is used, in the case of which the required contour should arrive at the point $f° \equiv f$ with coordinates $x_f = 1, y_f = 0$. It has already been noted that the contour which has been designed is only optimal when the inequalities (2.4) are satisfied at the point f. If, however, the first inequality of (2.4) is violated in the case of the contour then the OC contains a rear base, the size of which, that is, $y_{f°}$, is defined by the condition

$$[\Phi - \varphi + (\Phi_\vartheta + \Phi_p p_\vartheta)\sin\vartheta\cos\vartheta]_{f°} = 0 \tag{2.7}$$

On the other hand, if the second inequality of (2.4) is violated when the first is satisfied, then the optimal length of a body without a rear base, that is, the quantity $x_f < 1$, is defined, when $y_f = 0$, by the condition

$$(\Phi_\vartheta + \Phi_p p_\vartheta)_f \sin^2\vartheta_f = 0 \tag{2.8}$$

Although, under these conditions, one of the coordinates of a terminal point of the segment $if°$ or of the whole of the OC if, which are determined by Eq. (2.6), is unknown and is found using conditions (2.7) and (2.8), the second order of Eq. (2.6) ensures the arrival of the initial segment or of the whole of the required contour, respectively, at the point $f°$ or f. Finally, the choice of the constant undetermined Lagrange multipliers which occur in Eq. (2.6), at least, in principle, enables all of the isoperimetric conditions (1.9) to be satisfied.

3. THE OPTIMAL PROFILE FOR A SPECIFIED AREA OF THE LONGITUDINAL CROSS-SECTION

As an example, we consider a problem with a single isoperimetric condition, that is, a specified area of the longitudinal cross-section referred to the square of the length of the body

$$F = \int_{y_i=0}^{y_{f°}} (1-x)dy - \int_{y_f=0}^{y_{f°}} (1-x)dy \tag{3.1}$$

We note, in passing, that the area F simultaneously determines the volume per unit width of the profile.

Although the second term in (3.1) is equal to zero in the rear base where $x = 1$, in accordance with what has been said previously it is included in the formula for F in order to take account of the contribution to δC_x from a variation of the rear base in which the permissible $\delta x \leqslant 0$. In the example under consideration, we have

$$\Phi = p(\vartheta) + \mu(1-x), \quad \varphi = p^+ + \mu(1-x) \tag{3.2}$$

with a unique constant Lagrange multiplier μ. Substituting (3.2) into Eq. (2.6), we find that the segment $if°$ of the OC is defined by the equation

$$(p_\vartheta \sin^2\vartheta)' = \mu \tag{3.3}$$

In the case of a body without a rear base, conditions (2.4) take the form

$$[p(\vartheta) - p^+ + p_\vartheta \sin\vartheta\cos\vartheta]_f \geqslant 0, \quad (p_\vartheta \sin^2\vartheta)_f \geqslant 0 \tag{3.4}$$

In the local models considered above, the second inequality is always satisfied and becomes an equality in the case of the Newton model (1.3) when $\vartheta_f \leqslant 0$, and in the case of the other models only when $\vartheta_f = 0$.

Condition (2.7), which determines the optimal ordinate $y_{f°} > 0$ when the first inequality of (3.4) is violated in the case of a body without a rear base, is written as

$$[p(\vartheta) - p^+ + p_\vartheta \sin\vartheta\cos\vartheta]_{f°} = 0 \tag{3.5}$$

Finally, for a body with a rear base, the second inequality from (3.4) with f replaced by $f°$ together with the inequality obtained from (2.5) for (3.1)

$$\mu \leqslant 0 \tag{3.6}$$

are necessary conditions for the rear base to be a segment of a boundary extremum with respect to x.

In the case of the Newton model (1.3), Eq. (3.3), which determines the shape of i_{f°, takes the form

$$[(1+\operatorname{sign}\vartheta)\cos\vartheta\sin^3\vartheta]' = \mu \tag{3.7}$$

It immediately follows from this that, within the framework of this model, the OC cannot contain a leeward segment with negative $\vartheta > -\pi/2$ which is different from the rear base. Hence, when $F > 0$, the ordinate y_{f° is positive, $p \equiv p_\infty$ and the optimal body necessarily contains a rear base. The latter is clear, since, by virtue of (1.3), $\vartheta_{f^\circ} \geqslant 0$ on the leeward side of the contour independent of the magnitude of ϑ, while, on the windward side, p is greater, the greater the value of ϑ. Hence, within the framework of model (1.3), it is necessary to make the length of the windward side of the contour along x as large as possible ($x_{f^\circ} = 1$) which automatically reduces ϑ and p on it.

In fact, this result was obtained in those papers of the collection [6] in which plane and axially symmetric bodies of minimum drag were constructed within the framework of Newton's formula in cases where the half-height of a symmetric plane body or the radius of a body of revolution were not among the specified geometric characteristics. It is true, as has already been noted, that in the above-mentioned papers, having introduced a rear base, not only was it not shown that the rear base appears as a segment of a boundary extremum with respect to x but, in general, nothing was said about its existence.

Within the framework of model (1.3), the inequality $p^+ < p_\infty$ does not make sense which, incidentally, is also in agreement with condition (3.5) which, when $\vartheta_{f^\circ} \geqslant 0$, can only be satisfied for $p^+ \geqslant p_\infty$. It has already been mentioned that, in [6], p^+ has tacitly been put equal to p_∞. Unlike this, it is assumed below that $p^+ \geqslant p_\infty$. The latter may be the result of special effects of the type of heat supply to the base region and $p^+ > p_\infty$, when $M_\infty \gg 1$, for comparatively thick bodies and when there are no such effects [13]. Condition (3.5), which, when $p^+ \geqslant p_\infty$, yields a non-negative ϑ_{f°, takes the form, for $p(\vartheta)$ from (1.3)

$$\sin^2\vartheta_{f^\circ}(1+2\sin^2\vartheta\cos^2\vartheta)_{f^\circ} = N \equiv p^+ - p_\infty = \left(\frac{p^+}{p_\infty}-1\right)\frac{1}{\varkappa M_\infty^2} \tag{3.8}$$

where the second expression for N refers to a perfect gas.

Solving Eq. (3.8) for $q_{f^\circ} \equiv \operatorname{tg}\vartheta_{f^\circ}$, we find that

$$q_{f^\circ} = \left(\frac{3-\sqrt{9-8N}}{1+\sqrt{9-8N}}\right)^{1/2} \tag{3.9}$$

As in [2], Eq. (3.7) is integrated after which the parametric representation

$$\begin{aligned} x &= 1+\frac{1}{\mu}\left[\frac{1-q_{f^\circ}^2}{(1+q_{f^\circ}^2)^2}-\frac{1-q^2}{(1+q^2)^2}\right] \\ y &= y_{f^\circ}+\frac{2}{\mu}\left[\frac{q^3}{(1+q^2)^2}-\frac{q_{f^\circ}^3}{(1+q_{f^\circ}^2)^2}\right] \end{aligned} \tag{3.10}$$

is obtained for if°.

Here, $\mu \leqslant 0$ by virtue of (3.6), the parameter $q = \operatorname{tg}\vartheta$ decreases monotonically from $q_i > q_{f^\circ}$ to q_{f° from (3.9), q_i is expressed in terms of μ and q_{f° from the first equality of (3.10) with $x_i = 0$ and, after this, the multiplier μ is chosen in such a manner that the area of the longitudinal cross-section found using formulae (3.10) is equal to the specified magnitude of F. The segment of the contour if° is convex and, moreover, according to [2], $q \leqslant 1$. If q_i turns out to be greater than unity, then the OC contains two faces, $x = 1$ and $x = 0$, which corresponds to extremely large F.

In the case of the combined model (1.4)–(1.6), the windward segment of a contour with $\vartheta \geqslant 0$ is defined, as in the case of model (1.3), by Eq. (3.7). The leeward segment with $\vartheta \leqslant 0$ which is smoothly joined to it at the point $\vartheta = 0$, if it exists, satisfies the equation

$$(\rho V^2 \operatorname{tg}\alpha \sin^2 \vartheta)' = \mu \tag{3.11}$$

which, by virtue of (1.5) and (1.6) as well as (3.7), is a second-order differential equation in $x = x(y)$.

In the case of bodies with a sharp rear edge, where $\vartheta_f \leqslant 0$, the inequalities

$$(p - p^+ + \rho V^2 \operatorname{tg}\alpha \sin\vartheta \cos\vartheta)_f \geqslant 0, \quad (\rho V^2 \operatorname{tg}\alpha \sin^2 \vartheta)_f \geqslant 0 \tag{3.12}$$

must now be satisfied.

The second of these inequalities is always satisfied and becomes an equality only when $\vartheta_f = 0$. Unlike this, the first inequality can be violated even in the case when $p^+ = 0$, when, for a perfect gas

$$(p - p^+ + \rho V^2 \operatorname{tg}\alpha \sin\vartheta \cos\vartheta)_f = p_f \left(1 + \frac{\varkappa M^2 \sin 2\vartheta}{2\sqrt{M^2 - 1}}\right)_f$$

The second term in brackets, being negative in the case of a body without a rear base, increases in modulus as the Mach number M_f and the area F increase. Ultimately, its modulus necessarily becomes greater than unity. When $p^+ > 0$, the first condition in (3.12) is violated even earlier. From this instant y_{f° will be determined by the condition

$$(p - p^+ + \rho V^2 \operatorname{tg}\alpha \sin\vartheta \cos\vartheta)_{f^\circ} = 0 \tag{3.13}$$

and the conditions for $f^\circ f$ to be a segment of a boundary extremum with respect to x reduce to the second equality of (3.12), with f° instead of f, and to inequality (3.6). The model (1.7) differs from the combined model only in the fact that, in this model, Eq. (3.11) also determines the windward part of the contour. Finally, it can be shown that, in this problem, conditions (3.12) and (3.13) are identical to the necessary conditions for C_x to be a minimum which are obtained in the approximation of the Euler equations using a local variation [4] of the optimal contour in a small neighbourhood of the points f and f°.

In the case of a perfect gas, Eq. (3.11), by virtue of the equalities (1.5) and (1.6), takes the form

$$\frac{d^2 y}{dx^2} \equiv \ddot{y} = \frac{\mu \operatorname{ctg}\alpha}{2\rho V^2 \cos^4 \alpha} \zeta^{-1}, \quad \zeta = 1 + \frac{4(1 - M^2) + (\varkappa + 1) M^4}{4 \operatorname{ctg}^3 \alpha} \operatorname{tg}\vartheta \tag{3.14}$$

There is an expression which is linear with respect to x and x' and non-linear with respect to ϑ in the integral over if° of the Lagrange functional in the problem with a specified F. Hence, in model (1.7), for example, when terms of the second order in δC_x are retained, a single term is added

$$\delta C_x = \ldots + \int_0^1 \rho V^2 \zeta \operatorname{tg}\alpha (\delta\vartheta)^2 dx$$

Consequently, a further necessary condition for C_x to be a minimum in this model has the form

$$\zeta \geqslant 0, \quad x \in [0, 1] \tag{3.15}$$

By virtue of (3.14) and (3.15), the curvature of the OC does not change sign and is equal to zero everywhere when $\mu = 0$. In the case of optimal bodies without a rear base, it follows from this that, when $F \neq 0$, they are convex, since in the opposite case ($y \geqslant 0$), a contour lying above the x axis cannot join the points i and f which belong to this axis. It is seen from this that negative values of μ correspond to such bodies. For optimal bodies with a rear base, point i can be joined to the point f° at which $x_{f^\circ} = 1$ while $y_{f^\circ} > 0$ both by a convex curve and a concave curve. The fact that the segment if° of the OC is also convex here in model (1.7) and a straight line when $\mu = 0$ follows from (3.15) and (3.6). Hence, the necessary condition for the rear base to be a segment of a boundary extremum with respect to x plays an exceedingly important role here.

For $\mu < 0$, when the angle ϑ decreases monotonically on moving from i to f° or to f, it is convenient, as in model (1.3), to take $q = \operatorname{tg}\vartheta$ as the independent variable. In this case, Eq. (3.14) is replaced by two first-order equations which are integrated in quadratures

$$\frac{dx}{dq}=f(q)\equiv\frac{2\rho V^2\cos^4\vartheta}{\mu\,\mathrm{ctg}\,\alpha}\zeta,\quad \frac{dy}{dq}=qf(q)$$

Everything that has been said above for the leeward part (with $\vartheta < 0$) of the segment if° of the combined model is valid for the whole of the segment if° of the OC in model (1.7).

The equations and conditions, which determine the OC in the approximation of the linear model (1.8) and the equations and conditions (3.11)–(3.15) are obtained practically directly by linearization under the assumption that $|\vartheta| \ll 1$. In this case, the linearized equation (3.14), after integration, gives the equation of the contour if or if° in the form of a parabola

$$y = ax + bx^2 \tag{3.16}$$

with constants a and b which are still to be determined. When there is no rear base, this equation, which was obtained for the first time by Drougge [7], defines a parabola which is symmetric with respect to $x = 0.5$ with coefficients $a = -b = 6F$. It has already been shown by Chapman in [5] that the parabola (3.16) with such values of a and b, that is, the solution [7] without a rear base, only makes C_x a minimum for very small F even when $p^+ = 0$. For this reason, we shall refer to it as being "pseudo-optimal". The conditions under which it, nevertheless, ensures a minimum value of C_x are obtained by linearization of (3.12) and reduce to satisfying the two inequalities

$$\begin{aligned} &p_\infty - p^+ + \frac{\dot{y}_f}{\sqrt{M_\infty^2-1}} = p_\infty\left(1-\frac{p^+}{p_\infty}-\frac{12\varkappa M_\infty^2}{\sqrt{M_\infty^2-1}}F\right)\geqslant 0\\ &\dot{y}_f^2 = 36F^2 \geqslant 0 \end{aligned} \tag{3.17}$$

The second inequality of (3.17) is always satisfied. The first, however, as has already been noted, is violated for extremely small F. For instance, the minimum value of the coefficient of F, which is obtained when $M_\infty^2 = 2$, is equal to $24\varkappa$. When $M_\infty = \sqrt{2}$ and $\varkappa = 1.4$, according to [14], $p^+/p_\infty \approx 0.5$. It follows from this that, in the case of such M_∞ and $\varkappa$, the first inequality of (3.17) is already violated when $F \geqslant 0.015$, where the area has been made dimensionless by reference to the square of the length. At smaller and larger M_∞, it is violated in the case of even more slender profiles. Of course, a rear base, as was done for the first time in [5], can also be introduced within the framework of a linear model. The condition, defining the optimal size of the rear base, is obtained by linearizing (3.13) and, in the case of a perfect gas, reduces to the formula

$$\dot{y}_{f^\circ} = L \equiv \left(\frac{p^+}{p_\infty}-1\right)\frac{\sqrt{M_\infty^2-1}}{2\varkappa M_\infty^2} \tag{3.18}$$

As before, the segment if° of the OC remains the parabola (3.16). However, now

$$a = 3F - 0{,}5L,\quad b = 0{,}75(L - 2F) \tag{3.19}$$

Within the framework of the linear model, the optimal body has a rear base if, as follows from (3.17), the specified non-negative value of F exceeds the minimum value $F_m = -L/6$.

As in the case of the Newtonian model, for $p^+ \geqslant p_\infty$ when $F_m \leqslant 0$, the optimal body has a rear base for any $F > 0$.

Suppose that $p^+ = p_\infty$. Then, by virtue of (3.18) and (3.19), $L = 0$ and $a = -2b = 3F$. Then, using formulae (1.1), (1.8) and (3.16), we find that $\sqrt{(M_\infty^2 - 1)}C_x = 3F^2$ for the OC with a rear base in this case. A similar calculation in the case of a pseudo-optimal body without a rear base for which $a = -b = 6F$ gives $\sqrt{(M_\infty^2 - 1)}C_x = 12F^2$ in the case of the same contribution from its windward and leeward parts. Hence, the C_x values for pseudo-optimal bodies in the case of fixed F and $p^+ = p_\infty$ are greater by a factor of four than the C_x values for optimal bodies with a rear base. It is interesting that, if C_x is calculated for the same bodies using a linearized version of Newton's formula, the C_x values for a body without a rear base which are obtained are also four times greater. This result is obtained in spite of the different role of the leeward part of the body in the linear and the Newtonian models. According to (1.3), the whole of the contribution to C_x is solely made by the windward part of the body.

In the case of thin bodies, however, it is significant that the friction drag, which exceeds the wave drag or is comparable with it, almost independent of the form of the contour. Only its length is important and, in such cases, it is practically equal to the length of the body. Hence, the reduction in the drag of slender bodies with a rear base which has been found, as well as the results in [5], that were also calculated using the formulae of the linear theory, may turn out, in fact, to be not so impressive. The profiling of quite thick bodies has been carried out taking account of these considerations. In this case although the design itself was carried out within the framework of the Newtonian and linear models, the wave drag coefficient C_x for the bodies which had been designed was then calculated by numerical integration of the Euler equations using a monotonic second-order difference scheme with explicit construction of the low-shock wave.

Typical results of the calculations which were carried out are shown in Fig. 2 and in Table 1 for bodies with $F = \mathrm{tg}\,(30°)/6 = 1/(6\sqrt{3}) \approx 0.096$ in a flow of a perfect gas with $\varkappa = 1.4$ at different p^+/p_∞. The selected value of F corresponds to exceedingly thick bodies. For example, a pseudo-optimal body which is symmetric with respect to $x = 0.5$ and has a parabolic contour has $\vartheta_i = -\vartheta_f = 30°$ and a half-thickness $\tau_0 = y(0.5) \approx 0.144$ for such an F. In the case of bodies with a rear base, the cross-section of maximum thickness is attained for x which are either equal to or close to unity and a magnitude of τ close to τ_0.

The wave drag coefficients C_{xL}, C_{xN}, and C_{xD} for bodies with a rear base are given in Table 1. The optimal design of these bodies was accomplished within the framework of the linear and Newtonian models respectively, together with the design of the pseudo-optimal body without a rear base which was symmetric with respect to $x = 0.5$ and has been referred to above. Two values are given for C_x: that found numerically by integration of Euler's equations, which we refer to as the "exact" value, and (in brackets) that determined using the formulae of the Newtonian model for C_{xN} and the linear model in the case of C_{xL} and C_{xD}. In accordance with what has been said previously, Newtonian OCs were not constructed for $p^+/p_\infty = 0$. The relative differences between C_{xD} and C_{xL} are also given as percentages. These differences, like the C_x values themselves, were calculated using their exact and approximate values, that is, the values found using linear theory (the second of these is enclosed in brackets). Finally, the y_{f^*} values, which are optimal for the linear and Newtonian models, are given in the last two rows. For the p^+/p_∞ which have been considered, they depend weakly on the magnitude of the base pressure and increase as it increases. The effect of p^+/p_∞ decreases as the Mach number M_∞ increases. This is natural since $p_\infty/(\rho_\infty V_\infty^2) = 1/(\varkappa M_\infty^2)$ and, when $M_\infty \to \infty$, it tends to zero and the contribution to C_x, which is of the order of unity when p^+/p_∞, becomes much smaller than the contribution from the windward segment of the contour.

Figure 2, in which the contours for different values of M_∞ that are optimal within the framework of the linear model for different p^+/p_∞ (the numbers near the curves) and y/τ_0, are shown by the solid curves, demonstrates the weak dependence of p^+/p_∞ not only on y_{f^*} but also on the shape of the whole contour. The optimal contours (OCs) in the case of the Newtonian model when p^+/p_∞ are shown by the dashed lines. Furthermore, the contour of the pseudo-optimal body without a rear base, which is independent of M_∞ and p^+/p_∞, is represented in Fig. 2 by a solid line together with the OCs corresponding to $M_\infty = 3$.

A consideration of Table 1 and Figs 2 show that, in spite of the very large errors which are obtained in when determining C_x using the approximate models, the OCs designed using these models are close in shape and, especially, with respect to the exact values of C_x, which are also found to be significantly

Table 1

p^+/p_∞	0			1			2	
M_∞	3	6	12	3	6	12	6	12
$C_{xL} \times 10^4$	254	125	85	140	92	75	58	65
	(202)	(74)	(30)	(98)	(47)	(23)	(17)	(15)
$C_{xN} \times 10^4$				140	90	71	59	63
				(53)	(53)	(53)	(21)	(45)
$C_{xD} \times 10^4$	448	296	253	448	296	253	296	253
	(393)	(188)	(93)	(393)	(188)	(93)	(188)	(93)
$\Delta C_{xD}(\%)$	76	137	198	220	222	237	410	289
	(95)	(154)	(210)	(301)	(300)	(304)	(1005)	(520)
$y_{f^*L} \times 10^3$	116	130	137	144	144	144	159	151
$y_{f^*N} \times 10^3$				159	159	159	168	162

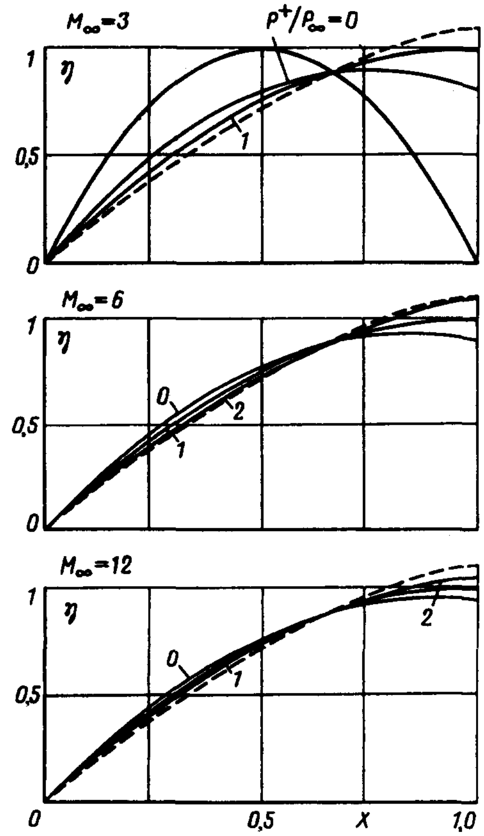

Fig. 2.

smaller than the exact values of C_x for pseudo-optimal bodies with a sharp rear edge. The closeness of the exact results for OCs designed using both models, even outside the range of their anticipated efficiency, is remarkable.

4. THE OPTIMAL CONFIGURATION WHEN $p^+ > p_\infty$

According to what has been said at the beginning of this paper, rear bases are not a unique feature of the problems under consideration. If $p^+ > p_\infty$, then hollow and partially hollow "check marks" are optimal. This section is concerned with such "non-standard" configurations.

As noted above, $p^+ > p_\infty$, first, in the case of sufficiently thick bodies when $M_\infty \gg 1$ and, second, when there are special actions in the base domain, such as a heat inflow in it, for example. Suppose that the second possibility is realized and, regardless of the magnitude of $F \geqslant 0$, $p^+/p_\infty > 1$ due to the additional action in the base domain. In this case, if the specified area F is reduced and approaches zero, then it would appear that concave extremals with the segment $ii°$ lying below the abscissa (Fig. 1c) can ensure the required small values of F. Such extremals are, however, forbidden for two reasons. First, as was established above, the OC in the problem under consideration cannot be concave and, second, $if°$ forms the upper part of a symmetric body and, hence, cannot pass below the abscissa. The other possible outcome which follows from this (Fig. 1d) in the form of a combination of the segment $ii°$ of the x axis with a convex extremal $i°f°$ and with a corner point where they join is rejected by the condition of transversality at the point $i°$, which does not permit a corner point.

The problem of designing the optimal contour when $p^+ > p_\infty$ and for sufficiently small F, which arises as a result, is solved in the following manner. We begin with a simpler problem without specifying F. The necessary conditions for C_x to be a minimum in the case of this problem are obtained from those found earlier if one puts $\mu = 0$ in them. Within the framework of any of the models which have been described above, we construct the OC which, when $\mu = 0$, will be a straight line and, from condition (3.5) with the corresponding expression for $p(\vartheta)$, we determine its optimal angle of inclination $\vartheta \equiv \vartheta_{f°}$ and the optimal $y_{f°} = \operatorname{tg} \vartheta_{f°}$. According to (3.9), in the case of the Newtonian model $y_{f°} = \operatorname{tg} \vartheta_{f°} = q_{f°}$

is a known function of the parameter N from (3.8) and, in the case of the linear model, $y_{f^\circ} = \operatorname{tg} \vartheta_{f^\circ} = L$ by virtue of (3.18). For any $\vartheta_{f^\circ} > 0$, the condition for an increase in C_x with a decrease in the length of the "body" is always satisfied. It is easy to see that the "check mark" that consists of the segment if°, which has been constructed in this way, and mirror reflection in the x axis provides the solution of the initial problem when $F = 0$. On filling the interior of the check mark in an arbitrary symmetric manner (Fig. 1e) we obtain the solution of the problem with a specified area F for any value $F \leqslant F_0$, where F_0 is the area of the completely filled check mark with a rear base ff°. The Lagrange multiplier μ becomes non-zero and negative and the contours if° become convex only when $F > F_0$.

Graphs of F_0 against N from (3.8) (the upper family of curves) and against L from (3.18) (the lower family), that is, against the similarity parameters which are obtained in this problem in the Newtonian and linear models as well as against M_∞ for a perfect gas with $\varkappa = 1.4$ are shown in Fig. 3. The curve calculated using the Newtonian model is represented by the dashed line and that calculated using the linear model is represented by the dot–dash line. The relations calculated using the exact formulae for a supersonic flow around a wedge with condition (3.13) at the end point f° of its contour are represented by the continuous curves. When there are no additional isoperimetric conditions, the contour of the optimal two-dimensional configuration is close to a straight line [4, 5] and, according to what has been said earlier, condition (3.13) holds at its end point and in the approximation of the complete system of equations of an ideal gas. Hence, the check marks which are obtained in this way (with the F_0 corresponding to them) can be considered as a solution of the same variational problem in a formulation which is close to the exact formulation. It can be seen from the behaviour of the upper family of curves in Fig. 3 that, as M_∞ increases, N becomes a similarity parameter not only in the Newtonian approximation but also in the almost exact approximation. Unlike this, L is not a similarity parameter.

As has already been noted, the extremal constructed in the Newtonian model (in this case it is a segment of a straight line) is optimal if, according to [2], $q \equiv \operatorname{tg} \vartheta \equiv \operatorname{tg} \vartheta_{f^\circ} \leqslant 1$ in it. By virtue of (3.9), $q_{f^\circ} = 1$ when $N = 1$. $F_0 = 0.5$ corresponds to this case and, by virtue of (1.3), $p^+ > 1 + p_\infty$, that is, the base pressure exceeds the magnitude corresponding to $\vartheta = \pi/2$. It can be shown that, for such p^+ and $F \leqslant 0.5$, a hollow or partially hollow check mark with $\operatorname{tg} \vartheta_{f^\circ} = 1$ and with a vertical screen f°, f^+ gives an optimal C_x in the Newtonian model which is negative here (thrust is created due to the large p^+). Its upper half is sketched in Fig. 1(f). In addition to the length of the body and $F \leqslant 0.5$, it is necessary in such cases to specify the maximum height of the construction Y and, moreover, as it is easy to comprehend, the optimal ordinate of the upper point of the screen $y_{f^+} = Y$. If $F > 0.5$, a front face ii°, in which $x = 0$, again appears in the case of the optimal body on account of the constraint on its length. In this case, the rectilinear segment of the contour $i^\circ f^\circ$ with $\vartheta = \pi/4$ is shifted upwards (Fig. 1g).

5. CRITICAL REMARKS ON THE DESIGN OF CLOSED OPTIMAL BODIES IN [5] AND [8]

The outstanding role played by Chapman's paper [5] has already been noted above. It considered, in the linear approximation, a whole class of problems on the design of two-dimensional bodies which are symmetric with respect to the $y = 0$ plane, sharpened in front and which realize a minimum in C_x for specified p^+, length and either maximum thickness τ in a previously unknown cross-section or the functional

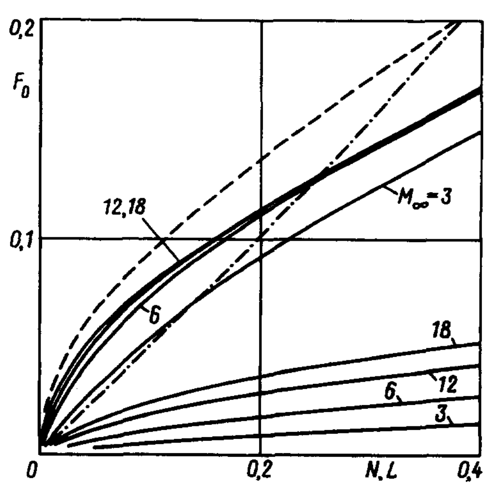

Fig. 3.

$$\chi = \frac{1}{\tau^\sigma}\int_0^1 y^n dx$$

with $n = 1, 2$ and 3 and $\sigma = 0$ and 1. If $n = 1$ and $\sigma = 0$, then $\chi = F$.

It has already been mentioned that a rear base was not treated as a segment of a boundary extremum in [5]. It is true that $\dot{y}_{f^\circ} < 0$ in the cases which were investigated in [5] with $p^+ < p_{f^\circ}$ by virtue of condition (3.18). The issue on the sign of the curvature of the extremal if° was therefore solved in [5] without invoking the inequality $\mu \leqslant 0$ that is one of the conditions for the optimality of a rear base as a segment of a boundary extremum which are missing from [5]. Moreover, in a problem with $\chi = F$, the negative curvature of if° that is obtained due to the sign of $\dot{y}_{f^\circ}$ necessitated the choice of negative μ which, in their turn, ensured that condition (3.6), missing from [5], is satisfied. The linear analogue of the second condition of (3.4), which, like (3.6), is also missing from [5], was also automatically satisfied when $\dot{y}_{f^\circ} < 0$. Hence, necessary conditions for a boundary extremum in the problems which have been solved in this paper, but have not been mentioned or verified in [5], could not have led to erroneous results. Unfortunately, what has been said did not guarantee that there would be no errors in [5] which are not associated with the rear base. In attempting to compensate for the increment $\Delta\tau$ which, when $\sigma = 1$, appeared in the expression for $\delta C_x = \delta I$ with $I = C_x + \mu\chi$, Chapman introduced, in the case of a free τ, a segment $y = \tau$ of length l° into the required contour and then varied it as a whole by putting $\delta y = \Delta\tau$ in it. As a result, the coefficient of $\Delta\tau$ was successfully made to vanish due to the choice of the "optimal" length l°. This, of course, is incorrect because the variations of y are free in the segment indicated. In fact, when $\sigma \neq 0$, the ordinate of the extremal if° attains a maximum equal to τ either at the point f° or at the internal corner point d. Taking account of this pact also gives the correct solution of problems with $\sigma \neq 0$.

The shortcomings which have been noted as well as the lack of any confirmation in [5] of the advantages with respect to C_x obtained in the linear approximation using more exact methods hardly explain the reason why the extremely efficient results in this paper were not published in the periodical press.

It has been emphasized more than once above that a failure to understand the reason for the appearance of a rear base is a shortcoming of almost all publications associated with the optimal design of closed bodies. In this sense, Large's paper [8], which has already been mentioned, is also extremely significant. In that paper, a rear base was introduced during the solution, in the Newtonian model approximation, of the design problem for an axially symmetric body which has a minimum wave drag coefficient C_x for a specified volume F and area Ω of the windward part of the surface, but where there are no constraints on the length and the radius. The "optimal" bodies with a rear base designed in [8] as a function of the dimensionless parameter $f = F/\Omega^{3/2}$ turned out to be sharpened or blunt (with $\dot{y}_i = \infty$), and their wave drag coefficient is positive and only vanishes when $f = 0$.

Without confining ourselves to the axially symmetric case, we show that the bodies designed in [8] are not optimal as, with a free length and a free half-height or radius in a problem with specified F and Ω, it is possible to design as many bodies as may be desired with $C_x \to 0$ and $C_D = C_x/\tau^{1+\nu} \to 0$. Henceforth, $\nu = 0$ in the two-dimensional case and $\nu = 1$ in the axially symmetric case, τ is the previously unknown half height or radius of the mid-section, all the linear dimensions, including the length of the body and τ are made dimensionless by reference to $\Omega^{1/(1+\nu)}$, and C_x by reference to Ω. When account is taken of the conversion to dimensionless quantities which has been adopted and the determination in the Newtonian model (1.3) of the "washed" surface, apart from unimportant positive factors, we have

$$1 = \frac{1}{2}\int_0^l y^\nu \sqrt{1+\dot{y}^2}(1+\operatorname{sign}\dot{y})dx, \quad f = \int_0^l y^{1+\nu} dx \tag{5.1}$$

$$C_x = \frac{1}{2}\int_0^l \frac{y^\nu \dot{y}^3}{1+\dot{y}^2}(1+\operatorname{sign}\dot{y})\, dx$$

where, as previously, the origin of the system of coordinates (of the cylindrical coordinates when $\nu = 1$) is placed at the leading point of the body.

Suppose that l_1 is the length of the windward part of the body, $l_2 = l - l_1$ is the length of the leeward part of the body and $y = \tau y_{1,2}(x_{1,2})$ are the equations of their contours with $x_1 = x/l_1$ and $x_2 = (x - l_1)/l_2$. In accordance with this, $\dot{y}_1 \equiv dy_1/dx_1 \geqslant 0$ and $\dot{y}_2 \equiv dy_2/dx_2 \leqslant 0$. Formulae (5.1) now take the form

$$1 = \tau^\nu l_1 k_1, \quad f = \tau^{1+\gamma}(l_1 k_2 + l_2 k_3), \quad C_x = \frac{\tau^{3+\nu}}{l_1^2} k_4$$

$$k_1 = \int_0^1 y_1^\nu \sqrt{1+\varepsilon\dot{y}_1^2}\, dx_1, \quad k_{2,3} = \int_0^1 y_{1,2}^{1+\nu} dx_{1,2} \tag{5.2}$$

$$k_4 = \int_0^1 \frac{y_1^\nu \dot{y}_1^3 dx_1}{1+\varepsilon\dot{y}_1^2}, \quad \varepsilon = \left(\frac{\tau}{l_1}\right)^2$$

where the first (second) subscript from the right correspond to the first (second) subscript in the formula for $k_{2,3}$.

We next select a quite arbitrary function $y_1(x_1)$ which increases monotonically from 0 to 1 and a function $y_2(x_2)$ which decreases monotonically from 1 to 0 when $x_{1,2} \in [0, 1]$ and we direct τ to zero. Then, by virtue of the first condition from (5.2) in the principal order with respect to τ as $\tau \to 0$, we shall have $l_1 = 1/(k_{10}\tau^{\nu})$. Henceforth, $k_{n0} = k_n$ when $\varepsilon = 0$. When account is taken of the expression obtained for l_1, we conclude that the combination of τ and l_1 occurring in formula (5.2) when $\tau \to 0$ behave in the principal orders with respect to τ as: $\varepsilon = k_{10}^2\tau^{2(1+\nu)} \to 0$, $\tau^{1+\nu}l_1 = \tau/k_{10} \to 0$ and $\tau^{3+\nu}/l_1^2 = k_{10}^2\tau^{3(1+\nu)} \to 0$. Hence, the first term in the formula for f as well as $C_x = k_{10}^2 k_{40}\tau^{3(1+\nu)}$ and $C_D = k_{10}^2 k_{40}\tau^{2(1+\nu)}$ simultaneously tend to zero. The specified value of f is preserved here if the length of the leeward part is taken as being equal to $l_2 = f/(k_{30}\tau^{1+\nu})$. The values of k_{n0} depend on the choice of the functions $y_{1,2}$ which gives an infinite set of bodies with C_x and C_D which tend to zero, that is, which are smaller than the finite C_x and C_D of the "optimal" bodies in [8].

6. ON THE DESIGN OF OPTIMAL BODIES OF A SPECIFIED LENGTH IN A VISCOUS GAS OR LIQUID STREAM

Approximate local models have been used in our investigation and, moreover, friction forces have been ignored. As far as the use of more accurate models is concerned, there is no need to expect any substantial changes of a fundamental nature in the case of a possible quantitative correction. As regards the effect of viscosity, it is necessary to distinguish the optimal design of bodies in a viscous supersonic flow and bodies in a flow of a viscous gas or a viscous incompressible liquid which does not give rise to even local supersonic zones. In the first case at high Reynolds numbers when friction forces can be calculated in the boundary-layer approximation, their addition to the wave drag while reducing the advantages (with respect to the total drag) of a body with a rear base compared with bodies with a sharp trailing edge does not affect the type of optimal configuration. This is due to the fact that, in such situations, the projection onto the x axis of the integral of the friction forces acting on the body, while being weakly dependent on the form of the contour, is determined mainly by its length. The existence of a boundary layer, as is well known [14], increases p^+ which, in its turn, leads to an increase in the size of the rear base. In the second case, when there is no wave drag, the issue on the advisability of introducing rear base requires further investigation. Nevertheless, in solving problems concerning the optimal design of bodies of fixed length in a flow of a viscous gas or a viscous incompressible liquid, the possibility of the appearance of a rear base in the optimal contour must necessarily be provided in advance. The results of experiments in [16] also confirm the fact that the introduction of a rear base can substantially improve the force characteristics of a body around which a flow occurs.

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REFERENCES

1. KRAIKO, A. N., The determination of bodies of minimum drag when using Newton's and Buzeman's drag laws. *Prikl. Mat. Mekh.*, 1963, **27**(3), 484–495.
2. KRAIKO, A. N., The leading part of a specified size which is optimal with respect to wave drag in the approximation of Newton's drag law. *Prikl. Mat. Mekh.*, 1991, **55**(3), 382–388.
3. KRAIKO, A. N., NAUMOVA, I. N. and SHMYGLEVSKII, Yu. D., The design of bodies of optimum form in a supersonic flow. *Prikl. Mat. Mekh.*, 1964, **28**(1), 178–182.
4. KRAIKO, A. N., *Variational Problems of Gas Dynamics*. Nauka, Moscow, 1979.
5. CHAPMAN, D. R., Airfoil profiles for minimum pressure drag at supersonic velocities—general analysis with application to linearized supersonic flow. *NACA Rep.*, 1952, 1063, 14.
6. MIELE, A. (ed.), *Theory of Optimum Aerodynamic Shapes*. Academic Press, New York, 1965.
7. DROUGGE, G., Two-dimensional wings of minimum pressure drag. In *Theory of Optimum Aerodynamic Shapes*, ed. A. Miele. Academic Press, New York, 1965, pp. 79–86.
8. LARGE, E., *Nonslender Nose Shapes of Minimum Pressure*, ed. A. Miele. Academic Press, New York, 1965, pp. 265–273.
9. GILBARG, D. and SHIFFMAN, M., On bodies achieving extreme values of the critical Mach number. I. *J. Ration. Mech. Analysis*, 1954, **3**(2), 209–230.
10. KRAIKO, A. N., Two-dimensional and axially symmetric configurations in a flow with maximum critical Mach number. *Prikl. Mat. Mekh.*, 1987, **51**(6), 941–950.
11. CHERNYI, G. G., *Gas Dynamics*. Nauka, Moscow, 1988.
12. LANDAU, L. D. and LIFSHITZ, E. M., *Fluid Mechanics*. Pergamon Press, Oxford, 1987.
13. YEL'KIN, Yu. G., NEILAND, V. Ya. and SOKOLOV, L. A., On the base pressure for a wedge in supersonic flow. *Inzh Zh.*, 1963, **3**(2), 362–366.
14. KORST, H. H., A theory for base pressure in transonic and supersonic flow. *J. Appl. Mech.*, 1956, **23**(4), 593–600.
15. KRAIKO, A. N. and PUDOVIKOV, D. Ye., The construction of the optimum contour of the leading part of a body in supersonic flow. *Prikl. Mat. Mekh.*, 1995, **59**(3), 419–434.
16. SATO, J. and SUNADA, Y., Experimental research on blunt trailing-edge airfoil sections at low Reynolds numbers. *AIAA Jnl*, 1995, **33**(11), 2001–2005.

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